Final Exam — Functional Analysis (WIFA–08)

Tuesday 4 April 2017, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 + 5 + 5 + 5 = 25 points)

Consider the following normed linear space:

$$X = \{f : [a,b] \to \mathbb{K} \; : \; f \text{ is bounded}\},$$
$$\|f\| = \sup_{x\in[a,b]} |f(x)|.$$

(a) Prove that X is a Banach space (i.e., every Cauchy sequence has a limit).

(b) Prove that $V = \{f \in X \; : \; f(a) = f(b) = 0\}$ is a linear subspace of X.

(c) Prove that V is closed in X.

(d) Compute $\dim(X/V)$.

Problem 2 (6 + 3 + 8 + 8 = 25 points)

Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| < 1$ and consider the following linear operator:

$$T : \ell^2 \to \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto (\alpha x_1, \alpha^2 x_2, \alpha^3 x_3, \dots).$$

Prove the following statements:

(a) $\|T\| = |\alpha|$;

(b) T is selfadjoint if and only if $\alpha \in \mathbb{R}$;

(c) T is compact;

(d) $\sigma(T) = \{\alpha^n \; : \; n \in \mathbb{N}\} \cup \{0\}$.

Problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) Formulate Baire's theorem for metric spaces.

(b) Let $\|\cdot\|$ be any norm on the space

$$\mathcal{P} = \{p : \mathbb{K} \to \mathbb{K} \,:\, p \text{ is a polynomial}\}.$$

Prove the following statements:

(i) $\mathcal{P}_n = \{p \in \mathcal{P} \,:\, \deg p \leq n\}$ is closed for each $n \in \mathbb{N} \cup \{0\}$;

(ii) $\mathcal{P}_n$ is nowhere dense for each $n \in \mathbb{N} \cup \{0\}$;

(iii) $\mathcal{P}$ is *not* a Banach space.

Problem 4 (5 + 5 + 3 + 7 = 20 points)

(a) Formulate the Hahn-Banach theorem for normed linear spaces.

(b) Let X be an infinite-dimensional normed linear space over $\mathbb{C}$. Pick $x_0 \in X$, $x_0 \neq 0$, and let $V = \operatorname{span}\{x_0\}$. Define the linear functional $f : V \to \mathbb{C}$ by setting $f(\lambda x_0) = (1 + 4i)\lambda\|x_0\|$.

Does there exist a functional $g \in X'$ such that $g \restriction V = f$ and:

(i) $\|g\| = 4$?

(ii) $\|g\| = \sqrt{17}$?

(iii) $\|g\| \geq 23$?

End of test (90 points)

Solution of Problem 1 (10 + 5 + 5 + 5 = 25 points)

(a) If (f_n) is a Cauchy sequence in X, then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
$$n, m \geq N \quad \Rightarrow \quad \|f_n - f_m\| \leq \varepsilon.$$
In particular, for each $x_0 \in [a, b]$ it follows that
$$n, m \geq N \quad \Rightarrow \quad |f_n(x_0) - f_m(x_0)| \leq \varepsilon, \tag{1}$$
which means that $(f_n(x_0))$ is a Cauchy sequence in $\mathbb{K}$.
(3 points)

Since $\mathbb{K}$ is complete the sequence $(f_n(x_0))$ converges. Hence, we can define $f : [a, b] \to \mathbb{K}$ by
$$f(x_0) = \lim_{n\to\infty} f_n(x_0), \qquad x_0 \in [a, b].$$
(2 points)

Taking $m \to \infty$ in the inequality (1) gives
$$n \geq N \quad \Rightarrow \quad |f_n(x_0) - f(x_0)| \leq \varepsilon,$$
and since $x_0 \in [a, b]$ is arbitrary it follows that
$$n \geq N \quad \Rightarrow \quad \|f_n - f\| = \sup_{x_0\in[a,b]} |f_n(x_0) - f(x_0)| \leq \varepsilon,$$
which means that $f_n \to f$ in X.
(3 points)

Finally, with $n = N$ it follows that $f_N - f \in X$ so that $f = f_N - (f_N - f) \in X$.
(2 points)

(b) If $f, g \in V$ and $\lambda, \mu \in \mathbb{K}$ then
$$(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a) = 0,$$
$$(\lambda f + \mu g)(b) = \lambda f(b) + \mu g(b) = 0,$$
which implies that $\lambda f + \mu g \in V$. This proves that V is a linear subspace of X.
(5 points)

(c) If $f \in \overline{V}$, then there exists a sequence $f_n \in V$ such that $f_n \to f$. Hence
$$|f(a)| = |f(a) - f_n(a)| \leq \|f - f_n\| \to 0,$$
which implies that $f(a) = 0$. Similarly, it follows that $f(b) = 0$. We conclude that $f \in V$ so that V is closed.
(5 points)

(d) Define the linear map $T : X \to \mathbb{K}^2$ by $Tf = (f(a), f(b))$. Then $\ker T = V$ and obviously $\operatorname{ran} T = \mathbb{K}^2$. Note that $X/\ker T \simeq \operatorname{ran} T$ which implies that $\dim(X/V) = 2$.
(5 points)

Solution of Problem 2 (6 + 3 + 8 + 8 = 25 points)

(a) Since $|\alpha| < 1$ it follows that $|\alpha|^n \leq |\alpha|$ for each $n \in \mathbb{N}$. Let $x \in \ell^2$ be arbitrary, then

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |\alpha^n x_n|^2 = \sum_{n=1}^{\infty} |\alpha|^{2n} |x_n|^2 \leq |\alpha|^2 \sum_{n=1}^{\infty} |x_n|^2 = |\alpha|^2 \, \|x\|^2,$$

which shows that

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \leq |\alpha|.$$

(4 points)

Note that for $x = (1, 0, 0, \dots)$ we have $\|x\| = 1$ and $\|Tx\| = |\alpha|$ which implies that $\|T\| = |\alpha|$.

(2 points)

(b) If $x, y \in \ell^2$ then

$$(Tx, y) = \sum_{n=1}^{\infty} \alpha^n x_n \bar{y}_n = \sum_{n=1}^{\infty} x_n \overline{\bar{\alpha}^n y_n} = (x, T^* y)$$

which shows that $T^* y = (\bar{\alpha} y_1, \bar{\alpha}^2 y_2, \bar{\alpha}^3 y_3, \dots)$. In particular, it follows that $T = T^*$ if and only if $\alpha \in \mathbb{R}$.

(3 points)

(c) Define for $k \in \mathbb{N}$ the operator

$$T_k : \ell^2 \to \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto (\alpha x_1, \dots, \alpha^k x_k, 0, 0, 0, \dots)$$

The same argument as in part (a) shows that T_k is bounded. In addition, $\operatorname{ran} T_k$ is finite-dimensional, which implies that T_k is compact.

(4 points)

Let $x \in \ell^2$ be arbitrary, then

$$\|(T - T_k)x\|^2 = \sum_{n=k+1}^{\infty} |\alpha|^{2n} |x_n|^2 \leq |\alpha|^{2k+2} \sum_{n=k+1}^{\infty} |x_n|^2 \leq |\alpha|^{2k+2} \|x\|^2.$$

which shows that $\|T - T_k\| \leq |\alpha|^{2k+2} \to 0$ as $k \to \infty$. Since each T_k is compact it follows that T is compact as well.

(4 points)

(d) Clearly, α^n is an eigenvalue of T for each $n \in \mathbb{N}$. The corresponding eigenvector is given by the n-th standard unit vector. Hence, $\{\alpha^n : n \in \mathbb{N}\} \subset \sigma(T)$.

(2 points)

Note that $\alpha^n \to 0$ since $|\alpha| < 1$. Since the spectrum is closed it follows that $0 \in \sigma(T)$ as well.

(1 point)

If $\lambda \notin \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$, then there exists $\delta > 0$ such that $|\lambda - \alpha^n| \geq \delta$ for all $n \in \mathbb{N}$. Note that

$$(T-\lambda)^{-1}x = \left(\frac{x_1}{\alpha - \lambda}, \frac{x_2}{\alpha^2 - \lambda}, \frac{x_3}{\alpha^3 - \lambda}, \dots \right)$$

so that

$$\|(T-\lambda)^{-1}x\|^2 = \sum_{n=1}^{\infty} \frac{|x_n|^2}{|\alpha^n - \lambda|^2} \leq \frac{1}{\delta^2} \sum_{n=1}^{\infty} |x_n|^2 = \frac{1}{\delta^2}\|x\|^2,$$

which shows that $(T-\lambda)^{-1}$ is bounded so that $\lambda \in \rho(T)$. Hence, $\sigma(T) = \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$.

(5 points)

Solution of Problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) Let X be a complete metric space and let $O \subset X$ be nonempty and open. Then O is nonmeager.

(5 points)

(b) (i) Note that $\mathcal{P}_n = \{p \in \mathcal{P} : \deg p \leq n\}$ is a finite-dimensional subspace of the normed linear space $\mathcal{P}$. This implies that $\mathcal{P}_n$ is closed.

(3 points)

(ii) We need to prove that $\operatorname{int} \overline{\mathcal{P}_n} = \emptyset$, or, equivalently, since $\mathcal{P}_n$ is closed, that $\operatorname{int} \mathcal{P}_n = \emptyset$.

(2 points)

If $p_0 \in \operatorname{int} \mathcal{P}_n$ then there exists $\varepsilon > 0$ such that

$$\{p \in \mathcal{P} : \|p - p_0\| < \varepsilon\} \subset \mathcal{P}_n.$$

Let $q \in \mathcal{P}$ be a nonzero polynomial and define $\widetilde{q} = p_0 + \frac{1}{2}\varepsilon q/\|q\|$ then

$$\|\widetilde{q} - p_0\| = \tfrac{1}{2}\varepsilon,$$

which implies that $\widetilde{q} \in \mathcal{P}_n$. In turn, this implies that

$$q = \frac{2\|q\|}{\varepsilon}(\widetilde{q} - p_0) \in \mathcal{P}_n$$

$q \in \mathcal{P}_n$ so that $\mathcal{P} = \mathcal{P}_n$, which is a contradiction. Hence, $\operatorname{int} \mathcal{P}_n = \emptyset$.

(5 points)

(iii) If $\mathcal{P}$ is a Banach space, then it is also a complete metric space. Since

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$$

it would follow from Baire's theorem that at least one of the sets $\mathcal{P}_n$ is *not* nowhere dense. This contradicts the conclusion of part (ii). Hence, we conclude that $\mathcal{P}$ is *not* a Banach space.

(5 points)

Solution of Problem 4 (5 + 5 + 3 + 7 = 20 points)

(a) Let X be a normed linear space and $V \subset X$ a linear subspace. For each $f \in V'$ there exists $F \in X'$ such that $F \upharpoonright V = f$ and $\|F\| = \|f\|$.

(5 points)

(b) (i) Note that

$$\|f\| = \sup_{\lambda \neq 0} \frac{|f(\lambda x_0)|}{\|\lambda x_0\|} = \sup_{\lambda \neq 0} \frac{|1+4i|\,|\lambda|\,\|x_0\|}{|\lambda|\,\|x_0\|} = |1+4i| = \sqrt{17}.$$

This means that if $g \in X'$ and $g \upharpoonright V = f$ then $\|g\| \geq \|f\| = \sqrt{17}$. This implies that there does not exist an extension g of f such that $\|g\| = 4$.

(5 points)

(ii) By the Hahn-Banach theorem there exists an extension $g \in X'$ of f such that $\|g\| = \|f\| = \sqrt{17}$.

(3 points)

(iii) Pick a nontrivial $x_1 \in X$ such that $\{x_0, x_1\}$ is a linearly independent set and define $g : \operatorname{span}\{x_0, x_1\} \to \mathbb{C}$ by setting

$$g(\lambda x_0 + \mu x_1) = (1+4i)\lambda\|x_0\| + 23\mu\|x_1\|. \tag{2}$$

Clearly, g is linear and $g \upharpoonright \operatorname{span}\{x_0\} = f$. Since g is defined on a *finite-dimensional* space it is automatically bounded. Note that $|g(\mu x_1)|/\|\mu x_1\| = 23$ for all $\mu \neq 0$ so that $\|g\| \geq 23$.

(5 points)

Now use the Hahn-Banach theorem to extend g to all of X while preserving the norm.

(2 points)